

Introduction to QPA

Part 4

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Outline

- 1 Examples of computations
- 2 Tilting modules

Ω -periodic modules

Λ – finite dimensional algebra

Recall: A module M is Ω -periodic if $\Omega_{\Lambda}^n(M) \simeq M$ for some positive integer n .

In some situations the period can indicate the degree of generators in the Hochschild cohomology ring,

$$\bigoplus_{i \geq 0} \text{Ext}_{\Lambda^{\text{env}}}^i(\Lambda, \Lambda).$$

Periodic algebras

Λ – finite dimensional k -algebra

$\Lambda^{\text{env}} = \Lambda^{\text{op}} \otimes_k \Lambda$ – enveloping algebra

Recall: Λ is a *periodic algebra* if Λ is a Ω -periodic module, that is $\Omega_{\Lambda^{\text{env}}}^n(\Lambda) \simeq \Lambda$ as Λ^{env} -modules.

Facts:

- Λ is a selfinjective algebra.
- All modules are Ω -periodic.
- The Hochschild cohomology modulo nilpotent elements is isomorphic to $k[x]$, where the degree of x is the period.

Finding quivers

Recall: $\Lambda = kQ/I$ with I admissible is a basic algebra, that is,

$$\Lambda = \bigoplus_{i=1}^t P_i$$

with P_i indecomposable, then $P_i \not\simeq P_j$ for $i \neq j$.

Facts:

- Λ – finite dimensional algebra.
- $\text{rad } \Lambda = \langle \text{arrows} \rangle / I$.
- $\Lambda / \text{rad } \Lambda \simeq \frac{kQ/I}{\langle \text{arrows} \rangle / I} \simeq \text{linear span of vertices}$.
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$$\begin{aligned} \text{rad } \Lambda / \text{rad}^2 \Lambda &\simeq \frac{\langle \text{arrows} \rangle}{I} / \frac{\langle \text{arrows} \rangle^2}{I} \\ &\simeq \frac{\langle \text{arrows} \rangle}{\langle \text{arrows} \rangle^2} \\ &\simeq \text{linear span of arrows} \end{aligned}$$

Finding quivers

Λ – finite dimensional algebra with $\Lambda / \text{rad } \Lambda \simeq k^n$ for some n .

Algorithm:

- ① Lift a complete set of orthogonal idempotents from $\Lambda / \text{rad } \Lambda$ to a complete set of orthogonal idempotents in Λ , say $\{e_i\}_{i=1}^n$ – the vertices.
- ② Compute $e_i \text{rad } \Lambda / \text{rad}^2 \Lambda e_j$, find a basis and lift back to $e_i \text{rad } \Lambda e_j$ – the arrows from vertex i to vertex j .
- ③ Construct a quiver Q from this and a homomorphism $\varphi: kQ \rightarrow \Lambda$.
- ④ Find the kernel of φ .

Trivial extensions

Λ – finite dimensional algebra

$T(\Lambda) = \Lambda \oplus D(\Lambda)$ – *trivial extension*,

$$(\lambda, f) \cdot (\lambda', f') = (\lambda\lambda', \lambda f' + f\lambda)$$

- $\text{rad } T(\Lambda) = \text{rad } \Lambda \oplus D(\Lambda)$.
- $\text{rad}^2 T(\Lambda) = \text{rad}^2 \Lambda \oplus D(\Lambda) \text{rad } \Lambda + \text{rad } \Lambda D(\Lambda)$.
- $\frac{\text{rad } T(\Lambda)}{\text{rad}^2 T(\Lambda)} \simeq \frac{\text{rad } \Lambda}{\text{rad}^2 \Lambda} \oplus \frac{D(\Lambda)}{D(\Lambda) \text{rad } \Lambda + \text{rad } \Lambda D(\Lambda)}$
- $T(\Lambda)$ is a symmetric algebra, $\Gamma \simeq D(\Gamma)$ as bimodules.

AR-theory

Recall that a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is *almost split exact* if it is not split exact and

- (i) for any not splittable epimorphism $t: X \rightarrow C$ there is a homomorphism $t': X \rightarrow B$ such that $gt' = t$,
- (ii) for any not splittable monomorphism $s: A \rightarrow Y$ there is a homomorphism $s': A \rightarrow Y$ such that $s'f = s$.

AR-theory

Facts:

- C and A are indecomposable modules.
- $A \simeq D \operatorname{Tr} C$ and $C \simeq \operatorname{Tr} D(A)$.
- For any indecomposable non-projective module C and for any indecomposable non-injective module A , there is an almost split sequence ending in C and starting in A .
- An almost split sequence is a generator of the socle of $\operatorname{Ext}_{\Lambda}^1(C, D \operatorname{Tr}(C))$ as an $\operatorname{End}_{\Lambda}(C)$ -module.

APR-tilting

$\Lambda = kQ$ – hereditary, Q no oriented cycle and connected.

S simple projective module (and not injective)

Λ is a classical tilting module T :

- $\text{pd}_{\Lambda} T \geq 1$,
- $\text{Ext}_{\Lambda}^1(T, T) = (0)$,
- the number of indecomposable non-isomorphic summands in T is equal to the number of isomorphism class of simple modules of Λ .

$\Lambda = P \oplus S \longrightarrow T = P \oplus \text{Tr } D(S)$ – APR-tilting